Applicability of Wong Approximation in Heavy-Ion Fusion Cross Section

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Abstract

We investigate the applicability Wong formula for heavy-ion fusion cross sections. Though this well-known formula gives almost exact results for single-channel calculations of relatively heavy systems, it overestimates the cross section for light systems. Thus, it is interesting to know how heavy a system should be in order for the Wong formula to work We perform a systematic calculation of heavy-ion reactions from light to heavy nuclei and determine the deviation between exact and Wong approximation. The results show that the deviation decreases as the mass of the target nuclei increases and becomes saturate at around Z ~ 20 and A ~ 50 (Z, A= atomic and mass number of target nuclei). Furthermore, the deviations decrease more rapidly for heavier projectiles than that of lighter ones.

Key words: heavy-ion, fusion cross section, Wong formula

Introduction

Heavy-ion fusion reactions at low incident energies provided a good opportunity to study the quantum tunneling phenomena of many-particle systems. The fusion cross section at energy E is equivalent to the transmission cross section of the Coulomb barrier given by the standard formula:

$$\sigma_F(E) = \frac{\pi\hbar^2}{2\mu E}\sum_l (2l+1)\mathrm{P}_l(E) \tag{1}$$

where, $P_l(E)$is barrier transmission probability (or) fusion probability of an angular momentum *l*. Therefore, the problem of determining the fusion cross section reduced to that of obtaining the transmission probability. Hill-Wheeler introduced a potential barrier of the form of an inverted parabola.

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Using the parabolic approximation, Wong has derived an analytic expression of fusion cross section [P. Frobrich *et.al*].

Parabolic Approximation and Wong Formula

Near the barrier top any reasonable potential can be approximated by a parabola as follows,

$$V(r) \sim V_B - \frac{1}{2}\mu\omega^2(r - R_B)^2 \quad (2)$$

where V_B and R_B are the barrier height and position, respectively. Here μ is the reduced mass of the system and ω is the barrier "curvature" given by

$$\omega^2 = -\frac{V''(r)}{\mu};$$

for which the Schrodinger equation becomes

$$(-\frac{\hbar^2}{2\mu}\nabla^2 + V_B - \frac{1}{2}\mu\omega^2(r - R_B)^2)\psi = E\psi \quad (3)$$

Using the parabolic approximation, Wong has derived an analytic expression of fusion cross sections [K. Hagono et.al (2005)]. He assumed that (i) the curvature of the Coulomb barrier is independent of the angular momentum *l*, and (ii) the dependence of the penetrability on the angular momentum can be well approximated by the shift of the incident energy as

$$P_l(E) = P_0\left(E - \frac{l(l+1)\hbar^2}{2\mu r_B^2}\right) \quad (4)$$

If many partial waves contributed to fusion cross section, the sum in Eq.(1) can be replaced by an integral;

$$\sigma_F(E) = \frac{\pi}{k^2}\int_0^\infty dl\,(2l+1)P_l(E) \quad (5)$$

Changing the variable from *l* to *l(l+1)*, the integration can be explicitly carried out, leading to the so called Wong formula

$$\sigma(E) = \frac{\hbar\omega}{2E}R_B^2 \ln[1 + \exp(\frac{2\pi}{\hbar\omega}(E - V_B))]\,. \quad (6)$$

One can find that the fusion cross section is determined by three parameters; the barrier height V_B, the barrier radius R_B and the curvature $\hbar\omega$. At high energies above the Coulomb barrier, the exponential in the argument of the logarithm in expression (6) is much larger than unity [K. Hagino *et.al*]. This formula gives the classical expressions,

$$\sigma(E) = \pi\, r_B^2 \left(1 - \frac{V_B}{E}\right) \text{for } E > V_B, \tag{7}$$

while at low energies, the exponential term is small, then

$$\sigma(E) \approx r_B^2 \frac{\hbar\omega}{2E} \exp\left(\frac{2\pi}{\hbar\omega}(E - V_B)\right) \text{for } E < V_B\,. \tag{8}$$

Deviation of Cross Section between Exact Solution and Wong Approximation

According to the assumptions used in the formula, one can generally say that Wong formula works well just around the Coulomb barrier and breaks down below the barrier []. In addition, it is also found that Wong formula does not work for light systems such as $^{12}C + ^{12}C$, although it works for heavy systems, such as $^{12}C + ^{154}Sm$ and ^{12}C ^{238}U. Fig.3 and 4 compare the exact and Wong fusion cross sections for light, medium and heavy systems, $^{12}C + ^{12}C$, $^{12}C + ^{154}Sm$ and $^{12}C + ^{238}U$, respectively. Upper panel is in linear scale and lower panel is in logarithmic scale [K. Hagino *et.al* (2012), I. Angeli *et.al*, Ozawa *et al*.]. As can be seen in figures, the heavier the systems the better the agreement between two calculations is.

One interesting question is "From where Wong formula starts working?", i.e., how heavy a system should be in order for the Wong formula to work? To answer this question, it will be necessary to calculate and compare the exact and approximate fusion cross sections for light to heavy mass regions. For this purpose, a systematic calculation is performed in this work. Firstly, choose ^{4}He as a projectile and change a target from ^{12}C to ^{238}U. For each target nuclide, an isotope which has the largest natural abundance has been chosen. Akyuz-Winther [O. Akyuz *et.al*]potential is taken as nuclear part of nucleus-nucleus potential. This potential is parameterized as the following form:

$$V_N(r) = \frac{-V_0}{1+\exp\left(\frac{r-R_p-R_T}{a}\right)} \tag{9}$$

with the potential depth is $V_0 = 16\pi\,\gamma\,a\,\overline{R}$. In this expression, the reduced radius is $\overline{R} = \frac{R_T R_P}{R_T + R_P}$ with $R_i = 1.20\,A_i^{1/3} - 0.09\ fm.$ The diffuseness parameter is

$\frac{1}{a} = 1.17(1+0.53(A_P^{-1/3} + A_T^{-1/3}))\ fm^{-1}$ while the surface tension parameter is

$$\gamma = 0.95[1-1.8\,\frac{N_P - Z_P}{A_P}\,\frac{N_T - Z_T}{A_T}]\ MeV\ \ fm^{-2} \tag{10}$$

Using this potential, the exact fusion cross section which provides barrier height, barrier position and curvature of Coulomb barrier for *s*-wave are calculated. Using these barrier parameters, approximate fusion cross section can be calculated with Wong formula Eq. (7).

The deviation between exact and Wong calculation is evaluated by integrating the following quantity:

$$\Delta\sigma = \frac{\int \left|\sigma_{exact}(E) - \sigma_{Wong}(E)\right| dE}{\int \sigma_{exact}(E)\,dE} \tag{11}$$

The integration limit has been chosen as E_{min}= 0.9 V_b and E_{max} = 1.1 V_b. The same procedure is done for ^{12}C, ^{16}O and ^{20}Ne as projectiles.

Fig. 5(a) and (b) show the calculated results as a function of atomic number Z and mass number A respectively. The deviation between exact and Wong cross section is normalized with the exact fusion cross section so as to compare the results. As can be seen in the figure, apart from very light projectile ^{4}He, fusion cross sections with three other projectiles show that the deviation decreases as the mass of the target nuclei increases and becomes saturated at around Z ~ 20 and A ~ 50 (Z, A= atomic and mass number of target nuclei).Furthermore, the deviations decrease more rapidly for heavier projectiles than that of lighter ones.

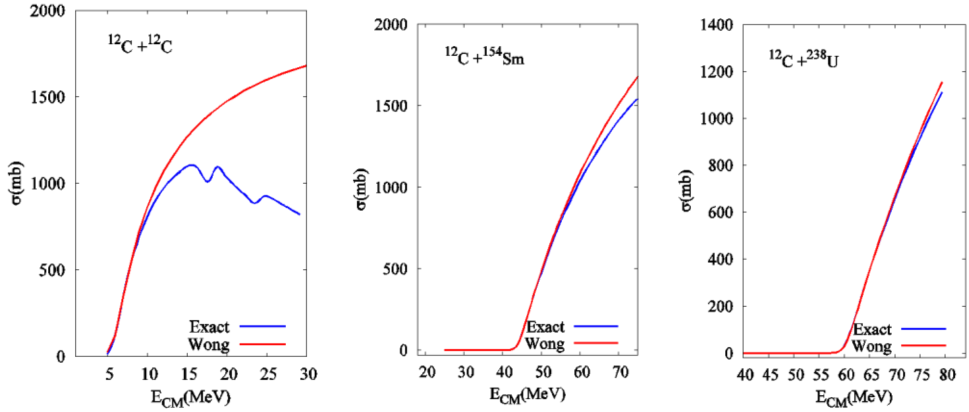

Fig. 3Comparison of exact and Wong fusion cross sections for light, medium nd heavy systems (^{12}C +^{12}C, ^{12}C +^{154}Sm and ^{12}C +^{238}U) in linear scale.

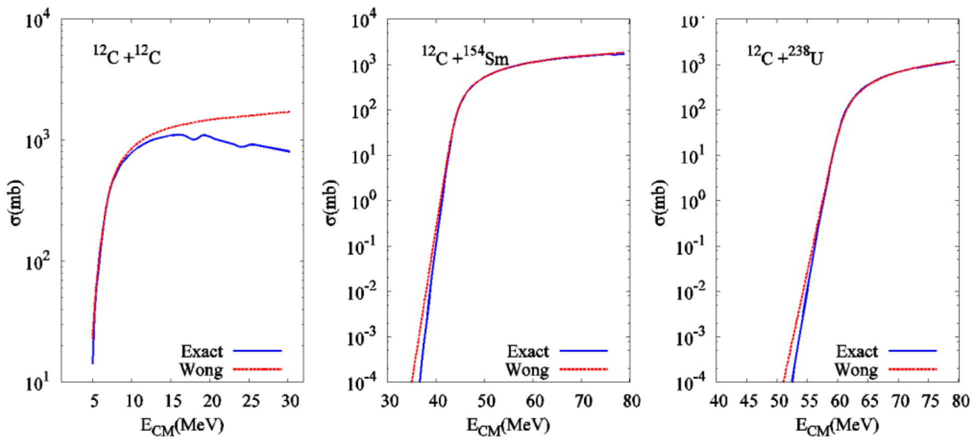

Fig. 4 Comparison of exact and Wong fusion cross sections for light, medium and heavy systems (^{12}C +^{12}C, ^{12}C +^{154}Sm and ^{12}C +^{238}U) in logarithmic scale.

For fusion reactions with the lightest projectile ^{4}He, the deviation shows relatively larger values than those of heavier projectiles. Although it shows decreasing behavior from light to heavy mass region, the deviation between exact and Wong fusion cross sections indicate rather large values than those of heavier projectiles even for heavier target region.

From this study, it can be concluded that Wong formula is not appropriate for fusion reactions with very light projectile like ^{4}He but will be applicable for reactions with other light projectiles like ^{12}C, ^{16}O and ^{20}Ne with target counterpart in the mass region around A ~50.

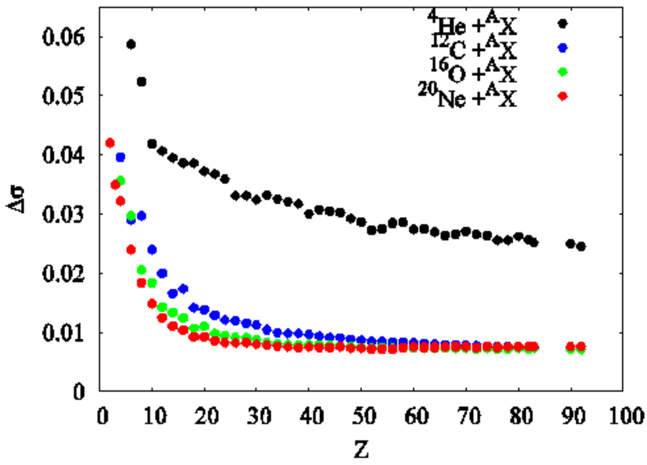

Fig. 5(a) Difference between exact and Wong fusion cross sections versus the atomic number of the target nuclei.

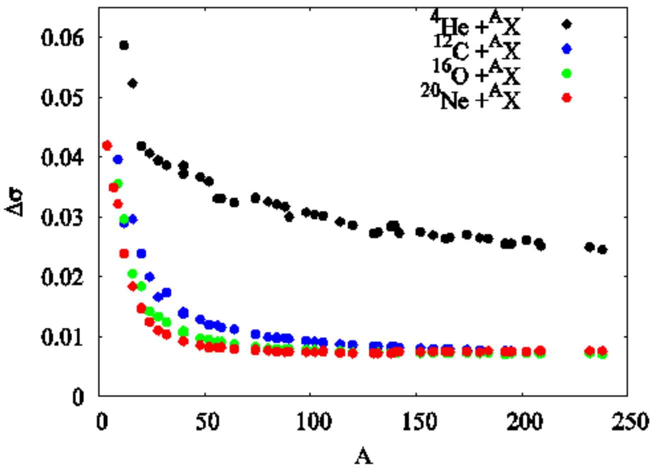

Fig. 5(b) Difference between exact and Wong fusion cross sections versus the mass number of the target nuclei.

Conclusions

The Wong formula has been widely used to estimate fusion cross sections for a given single-channel potential as well as to discuss the parameters which govern fusion. The formula is useful in discussing for example, the subbarrier enhancement of cross sections, providing reference cross sections in the absence of channel couplings. It is found that apart from very light projectile ^{4}He, fusion cross sections with three other projectiles show that the deviation decreases as the mass of the target nuclei increases and becomes saturated at around Z ~ 20 and A ~ 50 (Z, A = atomic and mass number of target nuclei).Furthermore, the deviations decrease more rapidly for heavier projectiles than that of lighter ones. Therefore, it can be concluded that Wong formula is not appropriate for fusion reactions with very light projectile like ^{4}He but will be applicable for reactions with other light projectiles like ^{12}C, ^{16}O and ^{20}Ne with target counterpart in the mass region around A ~50.

Acknowledgement

The authors thank Prof. Lei Lei Win, Head of Department of Physics, Mandalay University for her encouragement.

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